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ON THE STABILITY OF THE MOTION OF A VISCOELASTIC RING IN A GRAVITATIONAL FIELD*

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The motion in a circular orbit of a thin, homogeneous, unstretched viscoelastic ring in a central Newtonian gravitational field is investigated. An approximate non-linear system of differential equations is written out which describe the quasistatic state of motion of the ring relative to the centre of mass. A stability condition for the rotation of the ring in the plane of the orbit at an angular velocity which decays in magnitude is obtained in the first approximation. The asymptotic stability of the relative equilibrium of the ring in the orbital coordinate system when the ring lies in the plane of the orbit is proved, together with the instability of this equilibrium when the plane of the ring is perpendicular to the velocity vector of the centre of mass.

1. Let us consider the motion in a circular orbit of a homogeneous unstretched circular ring of constant cross-section in a central Newtonian gravitational field. The linear dimensions of the transverse cross-section of the ring will be assumed to be small compared with the radius r of its central line. We will denote the density of the ring by σ, F is the area of the transverse cross-section, EI is the flexural rigidity and $m = 2\pi rF\sigma$ is the mass of the ring.

Let $Ox_1x_2x_3$ be the coordinate system formed by the principal central axes of inertia of the undeformed ring with the x_3 axis perpendicular to the plane of the ring. The position of an element dm of the undeformed ring is specified by the angle α measured from the x_1 axis.

We shall consider those motions of the ring relative to the centre of mass when its elastic vibrations are flexural vibrations in the plane of the ring /1/. We will specify the position of an element dm of the undeformed ring by a radius vector ρ relative to the point O, where $\boldsymbol{\rho} = \boldsymbol{\xi} + \mathbf{u}$, where $\boldsymbol{\xi} = (x_1, x_2, x_3)$, $x_1 = r\cos\alpha$, $x_2 = r\sin\alpha$, $x_3 = 0$ and $\mathbf{u}$ is the elastic displacement. We shall represent the vector-function $\mathbf{u}(\boldsymbol{\xi}, t)$ in the form of a series in the orthonormalized characeristic modes of the free elastic vibrations of the ring. In the case of planar flexural vibrations, this series has the form**

$$\mathbf{u} = \sum_{n=2}^{\infty} q_n'(t)\,\mathbf{U}'^{(n)} + q_n''(t)\,\mathbf{U}''^{(n)} \tag{1.1}$$

$$\mathbf{U}'^{(n)} = \sqrt{\frac{2}{m(n^2+1)}}\begin{Vmatrix} n\cos n\alpha\cos\alpha + \sin n\alpha\sin\alpha \\ n\cos n\alpha\sin\alpha - \sin n\alpha\cos\alpha \\ 0 \end{Vmatrix} \tag{1.2}$$

$$\mathbf{U}''^{(n)} = \sqrt{\frac{2}{m(n^2+1)}}\begin{Vmatrix} n\sin n\alpha\cos\alpha - \cos n\alpha\sin\alpha \\ n\sin n\alpha\sin\alpha + \cos n\alpha\cos\alpha \\ 0 \end{Vmatrix}$$

In the case of free vibrations, the quantities q_n' and q_n'' satisfy the equations for harmonic vibrations with frequencies Ω_n which are given by the equalities

$$\Omega_n^2 = \frac{EI(n^2-1)^2 n^2}{\sigma F r^4 (n^2+1)} \qquad (n = 2, 3, \ldots) \tag{1.3}$$

The internal frictional forces which arise for arbitrary displacements of the elements of the ring will be simulated using a Rayleigh dissipative function of the type

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**Klimov D.M. and Markeyev A.P., Dynamics of an inhomogeneous elastic ring in a gravitational field, Preprint Inst. Problem Mekhaniki Akad. Nauk SSSR, 331, 1988.

$$R = \chi\beta \sum_{n=2}^{\infty} \Omega_n^2(\dot{q}_n'^2 + \dot{q}_n''^2), \quad \beta = \text{const} > 0 \tag{1.4}$$

where χ is a dimensionless parameter and differentiation with respect to time is indicated by a dot.

We shall specify the position of the ring in absolute space using the orbital coordinate system $OX_1X_2X_3$, the X_1, X_2 and X_3 axes of which are, respectively, directed along the transversal to the orbit, along the binormal and along the radius-vector of the centre of mass O with respect to the centre of attraction. We will specify the orientation of the coupled coordinate system $Ox_1x_2x_3$ relative to $OX_1X_2X_3$ using the Eulerian angles ψ, θ and φ.

A system of equations for the motion of a ring relative to the centre of mass has previously* been obtained which includes the equations of motion of the ring as a whole (the motion of the coordinate system $Ox_1x_2x_3$ relative to the orbital coordinate system) and the equations of the elastic vibrations of the body in the $Ox_1x_2x_3$ coordinate system.

Let us assume that the ring possesses a high rigidity and the decay of its free elastic vibrations occurs during a time which is much less than the period of rotation of the centre of mass along the orbit. Assuming that the latter quantity is of the order of unity, we introduce a small parameter ε by putting that the quantity ω_0/Ω_2 is of the order of ε (ω_0 is the average motion of the centre of mass). The above-mentioned assumption means that the inequality $0 < \chi \ll \varepsilon \ll 1$ is satisfied which enables one just to consider the quasistatic state of the motion of the ring /2/ over time intervals of the order of unity and greater. This quasistatic state corresponds to its forced elastic vibrations under the action of gravitational forces and inertial forces.

Under the assumption that $\chi \sim \varepsilon^\delta$ $(1 < \delta < 2)$, it is possible to obtain (see the second footnote) the following system of differential equations which describes the motion of the ring as a whole relative to the centre of mass in the quasistatic state of its elastic vibrations:

$$\begin{gathered}\psi'' \sin\theta + \psi'\theta' \cos\theta - \psi' \sin\psi \cos\theta - \theta' \cos\psi \sin\theta - (b + \varphi')\, c = \\ (\mu A_1 + {}^1/_3 \varkappa A_2) \sin 2\theta + {}^1/_9\, [(3\mu B_1 - 2\varkappa B_2)\, a - 3\mu B_3' + 2\varkappa B_4']\end{gathered} \tag{1.5}$$

$$\begin{gathered}\theta'' + \psi' \cos\psi - 3 \sin\theta \cos\theta + (b + \varphi')\, d = \sin\theta \cos\theta\, (\mu A_3 + {}^2/_3 \varkappa A_4) + \\ {}^1/_9 [(3\mu B_3 - 2\varkappa B_4)\, a + 3\mu B_1' - 2\varkappa B_2']\end{gathered} \tag{1.6}$$

$$\begin{gathered}\varphi' = -a' - {}^1/_9\, \varkappa\, [9\varphi' \sin^4\theta - 9 \sin^3\theta \cos\theta\, d - 3 \sin^2\theta a\, (c^2 - d^2) + \\ 6cd \sin\theta \cos\theta \quad \theta' + 3 \sin\theta \cos\theta\, d\, (c^2 + d^2) - b\, (c^2 + d^2)^2] \\ (\mu = 27\varepsilon^2\omega_0^2 / (10\lambda_2^2), \quad \varkappa = 81\chi\varepsilon^2\beta\omega_0^3 / (5\lambda_2^2))\end{gathered} \tag{1.7}$$

Quantities of the order of ε^4 and higher have been discarded in Eqs.(1.5)-(1.7), differentiation with respect to the variable $\tau = \omega_0 t$ is denoted by a prime and the following notation is adopted:

$$\begin{gathered}a = \psi' \cos\theta - \cos\psi \sin\theta, \quad b = \varphi' + a \\ c = \theta' + \sin\psi, \quad d = \psi' \sin\theta + \cos\psi \cos\theta\end{gathered} \tag{1.8}$$

$$\begin{gathered}A_1 = cd, \quad A_2 = 3 \sin^2\theta\varphi' - b\, (c^2 - d^2) - 3 \sin\theta \cos\theta d \\ A_3 = 3 \sin^2\theta + (c^2 - d^2), \quad A_4 = 2bcd - 3 \sin\theta \cos\theta\, (\theta' + c)\end{gathered} \tag{1.9}$$

$$\begin{gathered}B_1 = -c\, (3 \sin^2\theta + c^2 + d^2), \quad B_2 = A_2 d + A_4 c \\ B_3 = d\, (c^2 + d^2 - 3 \sin^2\theta), \quad B_4 = -A_2 c + A_4 d\end{gathered} \tag{1.10}$$

In (1.5)-(1.7) and subsequently, the notation $\Omega_2 = \varepsilon^{-1}\lambda_2$ is introduced for the lowest frequency of the planar free flexural vibrations of the ring. We note that, in quasistatic states, its planar flexural vibrations corresponding to the lowest frequency Ω_2 are the only substantial motions of the ring.

2. The system of Eqs.(1.5)-(1.7) has the following particular solutions:

$$\psi = \pi, \quad \theta = \pi/2, \quad \varphi' = \varphi_0' e^{-\varkappa\tau} \tag{2.1}$$

$$\psi = \pi, \quad \theta = \pi/2, \quad \varphi' = 0 \tag{2.2}$$

$$\psi = \pi/2, \quad \theta = \pi/2, \quad \varphi' = 0 \tag{2.3}$$

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In the case of (2.1), the ring is located in the plane of the orbit and rotates around the normal to the plane of the orbit at an angular velocity which decreases in magnitude.

The relative equilibrium of the ring in the orbital system of coordinates, when its plane lies in the plane of the orbit corresponds to solution (2.2).

In the case of (2.3), the ring is also located in an equilibrium position in the orbital coordinate system and the plane of the ring is perpendicular to the velocity vector of the centre of mass (the ring is arranged in a plane passing through the normal to the plane of the orbit and the radius-vector of the centre of mass of the ring relative to the centre of attraction).

The stability of the motions of the ring corresponding to solutions (2.1)-(2.3) is investigated below.

2.1. In the case of an absolutely solid ring ($\varepsilon = 0$) the motion (2.1) is stable when $\varphi_0' < -2$ or $\varphi_0' > -{}^1/_2$ and unstable when $-2 < \varphi_0' < -{}^1/_2$ /3, 4/. In the case of a visco-elastic ring ($\varepsilon \neq 0$, $\chi \neq 0$), we shall confine ourselves to an investigation in the first approximation. Let us put $\theta = \pi/2 + x, \psi = \pi + y, \varphi' = \varphi_0' e^{-\varkappa\tau} + z$. The linearized system of equations describing the perturbed motion will be

$$x'' + 4x + 2\varphi_0' e^{-\varkappa\tau}(1-\mu)(y'+x) - {}^4/_9\,\varkappa\varphi_0' e^{-\varkappa\tau}(4\varphi_0' e^{-\varkappa\tau}+1)(y - x') = 0 \tag{2.4}$$

$$y'' + y + 2\varphi_0' e^{-\varkappa\tau}(1+\mu)(y - x') - {}^4/_9\,\varkappa\varphi_0' e^{-\varkappa\tau}(4\varphi_0' e^{-\varkappa\tau}+1)(x + y') = 0 \qquad z' = -\varkappa z$$

We will now make use of the Lyapunov theorem on the stability of motion /5/ and select the Lyapunov function V in the following form (see the reference in the second footnote):

$$V = {}^1/_2\,(a_1x^2 + a_2y^2 + a_3x'^2 + a_4y'^2 + 2a_5x'y' + 2a_6xy + z^2) \tag{2.5}$$

$$a_1 = [2\varphi_0' e^{-\varkappa\tau} + 4 - 2\mu(\varphi_0' e^{-\varkappa\tau} + 4)]\,a_4, \quad a_2 = (2\varphi_0' e^{-\varkappa\tau} + 1 - 2\mu\varphi_0' e^{-\varkappa\tau})\,a_4$$

$$a_3 = (1 + 2\mu)\,a_4, \quad a_5 = a_6 = {}^2/_9\,\varkappa\,(4\varphi_0' e^{-\varkappa\tau} + 1)\,a_4$$

By virtue of the equations for the pertubed motion (2.4), its derivative V' has the form

$$V' = {}^1/_2\,(a_1'x^2 + a_2'y^2 + b_1x'^2 + b_2y'^2) - \varkappa z^2 \tag{2.6}$$

$$b_1 = (1 + 2\mu)\,a_4', \quad b_2 = a_4'$$

The function $a_4(\tau)$ can be taken as being arbitrary but such that the conditions $a_4 \geqslant \eta > 0$, $a_4' \leqslant 0$ ($\eta = \mathrm{const}$) are satisfied. If, in addition to that, constraints which are specified by the inequalities $a_1 > 0$, $a_2 > 0$, $a_1' < 0$, $a_2' < 0$ are imposed on the quantity φ_0', the function V will be positive definite and its derivative V' will also satisfy the requirements of the Lyapunov stability theorem by virtue of the constancy of its sign, which is opposite to the sign of V.

For example, let us put $a_4 = \varkappa + e^{-\tau}$. Then, an analysis of the above-mentioned inequalities leads to the following sufficient condition for the stability of the motion (2.1) for small values of the parameters ε and χ:

$$\varphi_0' > -{}^1/_2 + {}^1/_2\,(\mu + \varkappa) \tag{2.7}$$

2.2. To investigate the stability of the motion (2.2), we put $\theta = \pi/2 + x, \psi = \pi + y, \varphi' = z$. The equations for the perturbed motion can then be represented in the following form:

$$x'' = -4x + xy^2 + 2x^2y' + xy'^2 + {}^8/_3x^3 - 2z\,(x + y' - {}^1/_2xy^2 - {}^1/_6x^3 - {}^1/_2x^2y') + {}^1/_9\,(3\mu f_1 + 2\varkappa g_1) + O_4 \tag{2.8}$$

$$y'' = -y + {}^2/_3y^3 + 2xyy' + 2z\,(-y + x' + {}^1/_6y^3 - {}^1/_2x^2y + {}^1/_2x^2x') + {}^1/_9\,(3\mu f_2 + 2\varkappa g_2) + O_4 \tag{2.9}$$

$$z' = -xy + yy' + xx' + x'y' - 2z\,(xy - xx') - 2\mu z\,(xy - xx') - {}^1/_3\varkappa\,[3z\,(1 - 2x^2) + 4x^2 - y^2 + 2x'y + 5xy' - x'^2] + O_4 \tag{2.10}$$

Here, O_4 is the set of terms of not less than the fourth power with respect to x, y, x', y', z and, we denote by f_i, g_i ($i = 1, 2$) the following third-degree polynomials:

$$f_1 = 6z\,(x + y') + 8xy^2 + 2y^2y' - 22xyx' - 4x'yy' + 8x^3 + 18x^2y' + 12xy'^2 + 14xx'^2 + 2y'^3 + 2x'^2y'$$

$$g_1 = 6z^2\,(y - x') + 2y^3 - 9y^2x' - 19x^2y + 2yy'^2 + 4xyy' + 15x'^2y + 28x^2x' - 10xx'y' - 5x'y'^2 - 8x'^3$$

$$f_2 = 6z(x' - y) + 2y^3 - 6x'y^2 + 2x^2y + 4xyy' + 2yy'^2 + 6x'^2y - 8x^2x' - 10xx'y' - 2x'y'^2 - 2x'^3$$

$$g_2 = 6z^2(x + y') - 11xy^2 - 2y^2y' + 22xyx' + 4x'yy' - 28x^3 - 45x^2y' - 15xy'^2 - 8xx'^2 + x'^2y' - 2y'^3$$

The problem of the stability of system (2.8)-(2.10) belongs to the critical case of one negative and two pairs of purely imaginary roots. In order to solve this problem we shall make use of the "information principle" in the theory of stability /5/.

Let us make the change of variables $z \to \xi$ in system (2.8)-(2.10) which anihilates terms of the second degree on the right-hand side of Eq.(2.10) which only contain the critical variables x, y, x' and y'. We will seek a change of variables in the form

$$z = \xi + u_{11}x^2 + 2u_{12}xy + 2u_{13}xx' + 2u_{14}xy' + u_{22}y^2 + 2u_{23}yx' + 2u_{24}yy' + u_{33}x'^2 + 2u_{34}x'y' + u_{44}y'^2 \tag{2.11}$$

After the change of variables (2.11), Eq.(2.10) must take the form

$$\xi' = -2\xi(xy - xx') - 2\mu\xi(xy - xx') - \varkappa\xi(1 - 2x^2) + O_4 \tag{2.12}$$

It can be verified that the quantities u_{ij} satisfy a certain system of linear equations. The solution of this system, with an error of the order of $(\chi\varepsilon^2)^2$, has the form (see the reference in the second footnote)

$$u_{11} = \frac{1}{4}, \quad u_{12} = 0, \quad u_{13} = \frac{19}{96}\varkappa, \quad u_{14} = \frac{1}{2}, \quad u_{22} = \frac{5}{12} \tag{2.13}$$
$$u_{23} = 0, \quad u_{24} = \frac{1}{24}\varkappa, \quad u_{33} = -\frac{1}{16}, \quad u_{34} = \frac{1}{3}\varkappa, \quad u_{44} = -\frac{1}{12}$$

After the change of variables (2.11), Eqs.(2.8) and (2.9) become:

$$x'' = -4x - 2\xi(x + y') + h_1 + {}^1/_3\mu F_1 + {}^2/_9\varkappa G_1 + O_4 \tag{2.14}$$
$$y'' = -y + 2\xi(-y + x') + h_2 + {}^1/_3\mu F_2 + {}^2/_9\varkappa G_2 + O_4$$

where

$$h_1 = \frac{1}{6}xy^2 - \frac{5}{6}y^2y' + \frac{13}{6}x^3 - \frac{1}{2}x^2y' - \frac{5}{6}xy'^2 + \frac{1}{8}xx'^2 + \frac{1}{6}y'^3 + \frac{1}{8}x'^2y' \tag{2.15}$$

$$h_2 = -\frac{1}{6}y^3 + \frac{5}{6}y^2x' - \frac{1}{2}x^2y + \frac{1}{6}yy'^2 + \frac{1}{8}yx'^2 + \frac{1}{2}x^2x' + 2xx'y' - \frac{1}{6}x'y'^2 - \frac{1}{8}x'^3$$

$$F_1 = 6\xi(x + y') + F_1^*, \quad G_1 = 6\xi^2(y - x') + G_1^* \tag{2.16}$$
$$F_2 = -6\xi(y - x') + F_2^*, \quad G_2 = 6\xi^2(x + y') + G_2^*$$

$$F_1^* = \frac{21}{2}xy^2 + \frac{9}{2}y^2y' - 22xyx' - 4yx'y' + \frac{19}{2}x^3 + \frac{51}{2}x^2y' + \frac{35}{2}xy'^2 + \frac{109}{8}xx'^2 + \frac{3}{2}y'^3 + \frac{13}{8}x'^2y'$$

$$G_1^* = 2y^3 - 9y^2x' + \frac{5}{4}yy'^2 + \frac{13}{4}xyy' + 15yx'^2 + \frac{391}{16}x^2x' - \frac{313}{16}xx'y' - 11x'y'^2 - 8x'^3$$

$$F_2^* = -\frac{1}{2}y^3 - \frac{7}{2}y^2x' + \frac{1}{2}x^2y - 2xyy' + \frac{5}{2}yy'^2 + \frac{51}{8}yx'^2 - \frac{13}{2}x^2x' - 4xx'y' - \frac{5}{2}x'y'^2 - \frac{19}{8}x'^3$$

$$G_2^* = -11xy^2 - \frac{11}{4}y^2y' + \frac{295}{16}xyx' - \frac{5}{4}yx'y' - 28x^3 - 45x^2y' - 15xy'^2 - \frac{71}{16}xx'^2 - 2y'^3 + 7x'^2y'$$

The "contracted" system, which is obtained from Eqs.(2.14), if one puts $\xi = 0$ in these, has the form

$$x'' = -4x + \varphi_1 + O_4, \quad y'' = -y + \varphi_2 + O_4 \tag{2.17}$$
$$\varphi_i = h_i + {}^1/_3\mu F_i^* + {}^2/_9\varkappa G_i^* \quad (i = 1, 2)$$

The problem of the stability of the "contracted" system (2.17) belongs to the critical case of two pairs of purely imaginary roots. In order to solve it we shall use Kamenkov's algorithm /6/. First, it is necessary to obtain the normal form /7/ of the system (2.17) up to terms of the third power inclusive and than to transform to the polar coordinates ρ_i, θ_i $(i = 1, 2)$. When there are no 1:2 and 1:3 resonances, the stability conditions are expressed in terms of the coefficients of the right-hand sides of the equations for ρ_1 and ρ_2.

Since there are no second-order terms in (2.17), the 1:2 resonance which is present in the problem being considered does not affect the structure of the normal form and the equations for ρ_1 and ρ_2 can be obtained in the following manner. Let us make the change of variables $x, y, x', y' \to \rho_1, \rho_2, \theta_1, \theta_2$ according to the formulae $y = \rho_1 \cos\theta_1$, $y' = -\rho_1 \sin\theta_1$, $x = \rho_2 \cos\theta_2$, $x' = -2\rho_2 \sin\theta_2$.

If the terms in O_4 are dropped from system (2.17), then, in the new variables, it can be written in the form

$$\rho_1' = -\varphi_2 \sin\theta_1, \quad \rho_2' = -{}^1/_2\varphi_1 \sin\theta_2, \quad \theta_1' = 1 - \rho_1^{-1}\varphi_2 \cos\theta_1, \quad \theta_2' = 2 - {}^1/_2\rho_2^{-1}\varphi_1 \cos\theta_2 \tag{2.18}$$

By averaging the right-hand sides of the first two equations of (2.18) with respect to θ_1 and θ_2, we obtain the required equations for ρ_1 and ρ_2:

$$\zeta_i' = \zeta_i (c_{i1}\zeta_1 + c_{i2}\zeta_2) \quad (i = 1, 2) \tag{2.19}$$
$$(\zeta_i = \rho_i^2, \quad c_{11} = -35\varkappa/72, \quad c_{12} = -17\varkappa/9, \quad c_{21} = -20\varkappa/9$$
$$c_{22} = -1145\varkappa/288)$$

According to the Kamenkov criterion, the "contracted" system (2.17) is asymptotically stable and this conclusion is independent of terms higher than the third order on its right-hand sides. Also, since the series expansion of the right-hand side of Eq.(2.12), calculated when $\xi = 0$, commences with terms of not less than the fourth order, it may be concluded on the basis of the "information principle" that the full system (2.8)-(2.10) and the motion (2.2) which is being considered are asymptotically stable.

We note that, in the case of an absolutely rigid ring, the motion (2.2) is simply Lyapunov stable /3, 8/.

2.3. In order to investigate the stability of the motion, which corresponds to the solution of (2.3), we put $\theta = \frac{\pi}{2} + x$, $\psi = \frac{\pi}{2} + y$, $\varphi' = z$. The linearized system of equations of the perturbed motion will be

$$x'' + 3(1 - {}^2/_3\mu)x + {}^2/_3\varkappa x' = 0 \tag{2.20}$$
$$y'' - (1 - {}^2/_3\mu)y - {}^2/_3\varkappa y' - (2 + {}^2/_9\varkappa)z = 0$$
$$z' + y' + {}^4/_9\varkappa(2z - y) = 0$$

The characteristic equation of system (2.20) has two pairs of complex conjugate roots

$$\lambda_{1,2} = -{}^1/_3\varkappa \pm i\sqrt{3}(1 - {}^1/_3\mu) + O(\varepsilon^4), \quad \lambda_{3,4} = -\varkappa \pm i(1 + \mu) + O(\varepsilon^4)$$

and a single real positive root

$$\lambda_5 = 16/9\varkappa + O(\varepsilon^4)$$

The relative equilibrium of the ring being considered is therefore unstable.

We note that, in the case of an absolutely rigid ring $(\varepsilon = 0)$, the solution of (2.3) corresponds to the special case to a hyperboloidal precession which is Lyapunov stable /8/ and this follows from the fixed-sign property of the energy integral in the neighbourhood of the solution of (2.3). The presence of internal viscosity $(\chi \neq 0)$ in the material of an elastic ring $(\varepsilon \neq 0)$ destroys this stability and the motion becomes unstable.

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SLIPPING REGIMES IN MECHANICAL SYSTEMS*

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Mechanical systems with non-bilateral kinematic constraints are considered. For such systems conditions are obtained for which their equations of motion, determined by methods from classical mechanics, convex underdetermination and equivalent control, are identical.

Problems associated with methods of obtaining the equations of motion in slipping regimes, appearing in systems of differential equations with discontinuous right-hand sides, have been most thoroughly discussed in /1, 2/. From the point of view of classical mechanics, the appearance of such regimes amounts to imposing on a system of material points $P_i, (i = 1, 2, \ldots, N)$ some non-bilateral relations S_β $(\beta = 1, 2, \ldots, m)$

$$S_\beta = \varphi_\beta(t, r, r^\cdot), \; r = (r_1, \ldots, r_N), \; r^\cdot = (r_1^\cdot, \ldots, r_N^\cdot) \tag{0.1}$$

where r_i and $r_i^\cdot$ are respectively the position vectors and velocities of the points P_i.

In this context slipping motion corresponds to motion in which the constraints S_β become bilateral during certain modes of behaviour, i.e. $S_\beta = 0$. Then the right-hand sides of the dynamical equations for the points P_i undergo discontinuities on the hypersurfaces $S_\beta = 0$.

Below we shall assume that when the links are bilateral (embedding), they are ideal. This assumption is natural for a wide class of mechanical systems.

For example, such a situation occurs when S_β is a non-bilateral frictional constraint /3/. If the constraint (0.1) is ideal (in the above sense) and linear in the velocities, then to derive the equations of motion for the points P_i of a constrained system one can apply Lagrange's method of undetermined multipliers (alternatively, the method of convex underdetermination and equivalent control /1, 2/). However, any one of these approaches in isolation may not give sufficient information for investigating the behaviour of such systems with variable structure. In particular, the Lagrange method, uniquely defining the slipping equations, does not, in general, establish their switching conditions, whereas the methods in /1, 2/, in principle giving conditions for the existence of singular regimes, in a range of cases do not guarantee the correctness of the derivation of the equations of motion when the constraints S_β are bilateral.

In connection with these and other problems there is the interesting problem of the consistency of the various methods of deriving equations of motions for slipping regimes for mechanical systems of variable structure within the framework of Newtonian mechanics.

1. We will first consider a dynamical system of the form

$$m_i x_{ij}^{\cdot\cdot} = F_{ij}(t, x, x^\cdot) + \sum_{\beta=1}^{m} b_{i\beta}^j(t, x, x^\cdot)\, u_\beta(x, x^\cdot) \tag{1.1}$$

$$u_\beta = \begin{cases} u_\beta^+, S_\beta > 0, \\ u_\beta^-, S_\beta < 0, \end{cases} \quad S_\beta = \sum_{i, \; =1}^{N, 3} l_{\beta i} x_{ij}^\cdot + l_{\beta(N+1)}^4 \tag{1.2}$$

Prikl. Matem. Mekhan., 55,1,26-31,1991